Ex 1 Does $\sum_{n=0}^{\infty} \frac{2^n}{n!}$ converge or diverge?

(a) Converges

(b) Diverges

$$0 \le \frac{2^n}{n!} = \frac{2}{n} \cdot \frac{2}{n-1} \cdot \frac{2}{n-2} \cdots \frac{2}{1} \le \frac{2}{n} \cdot \frac{2}{n-1} \cdot 2$$

$\le 1 \quad \le 1 \cdots \le 1 \quad \le 2$

$$= \frac{8}{n(n-1)} \quad \text{for } n \ge 3$$

But $\sum_{n=1}^{\infty} \frac{8}{n(n-1)}$ converges (4th L.C.T.), so

$\sum_{n=0}^{\infty} \frac{2^n}{n!}$ also converges

Q. Why can't we use the integral test to determine the behavior of the above series?

No such thing as $\frac{2^x}{x!}$

(ok, there is. But

(1) too involved

(2) sledge hammer)

Q. How do we work with factorials?

Q. Consider the sequence defined by $a_n = r^n$ for $n \geqslant 0$ and the corresponding series $\sum_{n=0}^{\infty} a_n$.

(a) Under what conditions does the series converge?

$|r| < 1$

(b) What is the ratio $\frac{a_{n+1}}{a_n}$?

$$\frac{a_{n+1}}{a_n} = \frac{r^{n+1}}{r^n} = r$$

(c) Rewrite the conditions from (a) to make a statement like "$\sum_{n=0}^{\infty} a_n$ converges if and only if $\frac{a_{n+1}}{a_n}$..."

$$\left| \frac{a_{n+1}}{a_n} \right| < 1$$

Q. What is $\frac{(n+1)!}{n!}$?

$$\frac{(n+1)\cdot n\cdot(n-1)\cdots 2\cdot 1}{n\cdot(n-1)\cdots 2\cdot 1} = n+1$$ nice, cts.

Thm: (The Ratio Test) Suppose $\sum_{n=1}^{\infty} a_n$ is a series with nonzero terms and suppose $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = r$.

Then

(a) If $r>1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

(b) If $0<r<1$, then $\sum_{n=1}^{\infty} a_n$ converges.

(c) If $r=1$, then the test is inconclusive.

Ex 2 Determine the behavior of $\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!}$.

$$a_n = \frac{(n!)^3}{(3n)!}$$

$$a_{n+1} = \frac{((n+1)!)^3}{(3(n+1))!}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\left[\frac{((n+1)!)^3}{(3(n+1))!}\right]}{\left[\frac{(n!)^3}{(3n)!}\right]} = \lim_{n\to\infty}\frac{(3n)!}{(3n+3)!}\frac{((n+1)!)^3}{(n!)^3}$$

$$= \lim_{n\to\infty}\frac{3n\cdot(3n-1)\cdot(3n-2)\cdots 1}{(3n+3)(3n+2)(3n+1)\cdot 3n\cdot(3n-1)\cdots 1}\cdot\left(\frac{(n+1)\cdot n\cdot(n-1)\cdots 1}{n\cdot(n-1)\cdots 1}\right)^3$$

$$= \lim_{n\to\infty}\frac{(n+1)^3}{(3n+3)(3n+2)(3n+1)} = \frac{1}{27} < 1$$

By the ratio test, $\sum_{n=1}^{\infty}\frac{(n!)^3}{(3n)!}$ converges

Ex 3 Use the ratio test to determine the behavior of $\sum_{n=1}^{\infty} \frac{2n^3+7n-50}{(-2)^n}$.

(a) Converges
(b) Diverges

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\left(\frac{2(n+1)^3+7(n+1)-50}{(-2)^{n+1}}\right)}{\left(\frac{2n^3+7n-50}{(-2)^n}\right)}\right|$$

$$= \lim_{n\to\infty} \frac{2^n}{2^{n+1}} \frac{2(n+1)^3+7(n+1)-50}{2n^3+7n-50}$$

$$= \frac{1}{2} < 1$$

By the ratio test, $\sum_{n=1}^{\infty} \frac{2n^3+7n-50}{(-2)^{n-1}}$ converges

Q. For what types of series is the ratio test helpful?

- factorials
- # fn. of n

Ex 4 What happens if we try to use the ratio test on p-series?

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^p}}{\frac{1}{n^p}} = \lim_{n\to\infty} \frac{n^p}{(n+1)^p} = 1$$

$\Rightarrow$ inconclusive...

Ex 5 Does $\sum_{n=1}^{\infty} \frac{n^{3n}}{(2n+7)^n}$ converge or diverge?

(a) Converges

(b) Diverges

Ratio test attempt:

$$\lim_{n\to\infty} \frac{\frac{(n+1)^{3(n+1)}}{(2(n+1)+7)^{n+1}}}{\left(\frac{n^{3n}}{(2n+7)^n}\right)} = \lim_{n\to\infty} \frac{(2n+7)^n}{(2(n+1)+7)^{n+1}} \cdot \frac{(n+1)^{3(n+1)}}{n^{3n}}$$

$= ???$

L'Hôpital is legal, but not helpful

Q. Consider the sequence defined by $a_n = r^n$ for $n \geqslant 0$ and the corresponding series $\sum_{n=0}^{\infty} a_n$.

(a) Under what conditions does the series converge?

$|r| < 1$

(b) What is the root $\sqrt[n]{a_n} = |a_n|^{1/n}$? (and why is the absolute value necessary?)

$|a_n|^{1/n} = |r^n|^{1/n} = |r|$

(c) Rewrite the conditions from (a) to make a statement like "$\sum_{n=0}^{\infty} a_n$ converges if and only if $|a_n|^{1/n}$..."

$|a_n|^{1/n} < 1$

Thm: (The Root Test) Suppose $\sum_{n=1}^{\infty} a_n$ is a series with $\lim_{n\to\infty} |a_n|^{1/n} = r$.

(a) If $r > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

(b) If $r < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

(c) If $r = 1$, then the test is inconclusive.

Ex 6 Determine the behavior of $\sum_{n=1}^{\infty} \frac{n^{3n}}{(2n+7)^n}$.

$$\lim_{n\to\infty} \left| \frac{n^{3n}}{(2n+7)^n} \right|^{1/n} = \lim_{n\to\infty} \frac{n^{3n/n}}{(2n+7)^{n/n}} =$$

$$= \lim_{n\to\infty} \frac{n^3}{(2n+7)} = \infty > 1$$

By the root test, $\sum_{n=1}^{\infty} \frac{n^{3n}}{(2n+7)^n}$ diverges

Ex 7 Determine the behavior of $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$.

(a) Converges
(b) Diverges

$$\lim_{n\to\infty} \left|\left(\frac{n}{n+1}\right)^{n^2}\right|^{1/n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{n}$$

$$= \lim_{n\to\infty} \frac{1}{\left(\frac{n+1}{n}\right)^{n}}$$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^{n}} = \frac{1}{e} < 1$$

Ex 8 Determine the behavior of $\sum_{n=2}^{\infty} (\ln(n))^n$.

(a) Converges
(b) Diverges

Q. For what types of series is the root test helpful?

$$\left(\text{fn of } n\right)^{\text{fn. of } n}$$